The zero forcing number of Hamming graphs

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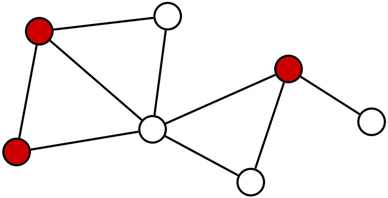
28 March 2024

Joint work with Aida Abiad and Sjanne Zeijlemaker

Zero forcing

"If a red vertex has a unique white neighbor, color it red."



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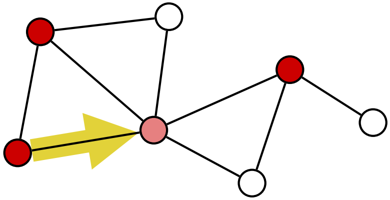

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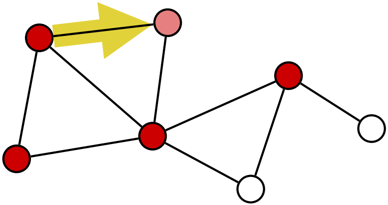

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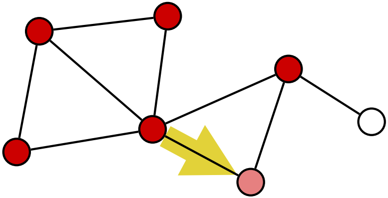

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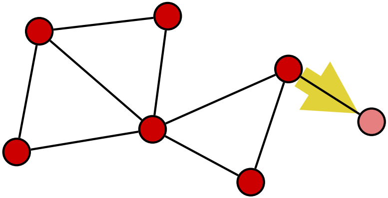

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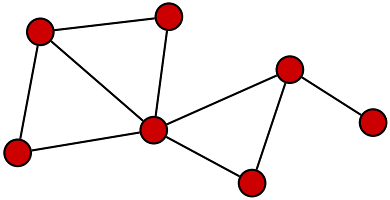

$Z(G) =$ the smallest size of a set that colors all vertices red

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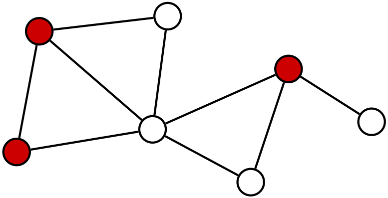

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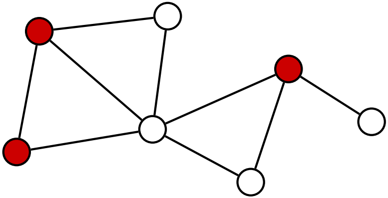

$Z(G) = 3$

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Applications of zero forcing

- Recommender systems
- Electric power networks
- Control of quantum systems

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Definition

The *Hamming graph* $H(n, q)$ has vertex set $\{0, \ldots, q-1\}^n$, where two vertices (n-tuples) are adjacent if they differ in one position.

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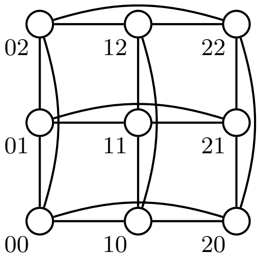
- $H(2,q)$ is the *square lattice* $K_q \times K_q$.



Hamming graphs

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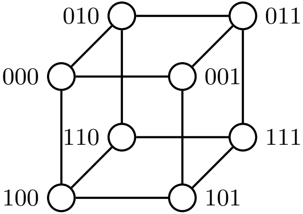
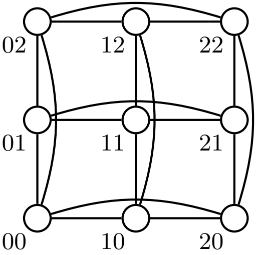
- $H(2,q)$ is the *square lattice* $K_q \times K_q$.
- $H(n,2)$ is the *hypercube* Q_n.

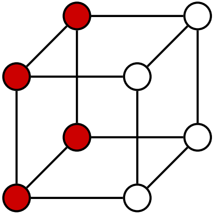




Hypercubes

Theorem (AIM Minimum Rank - Special Graphs Work Group, 2008 and Alon, 2008, independently)

$$Z(Q_n) = 2^{n-1}$$



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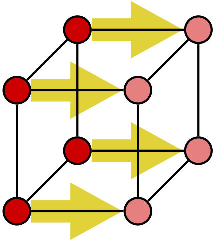
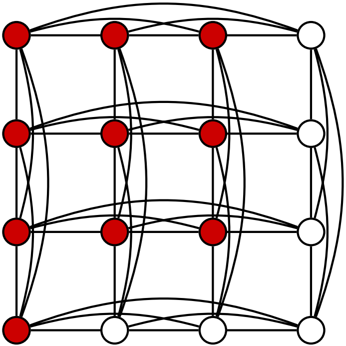


Theorem (AIM Minimum Rank - Special Graphs Work Group, 2008)

$$Z(H(2,q)) = q^2 - 2q + 2$$



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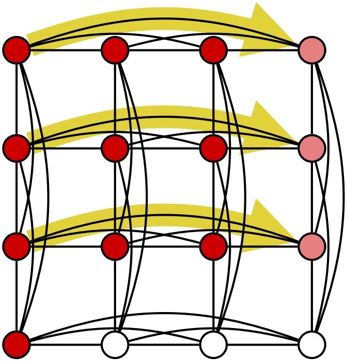

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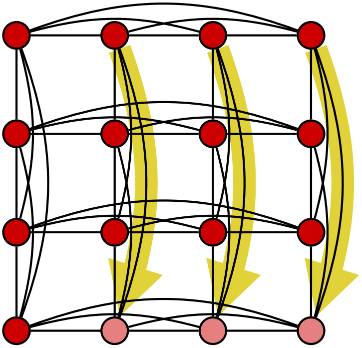
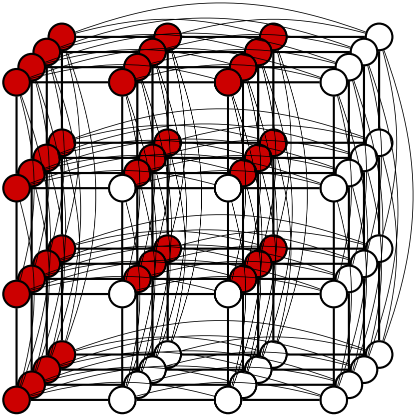
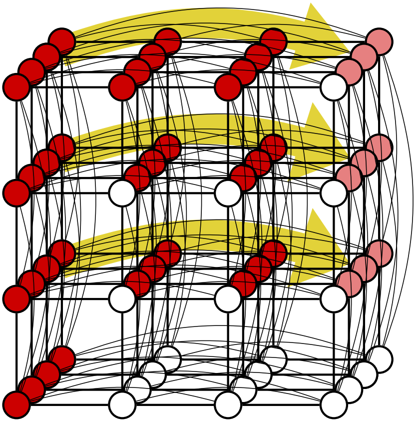
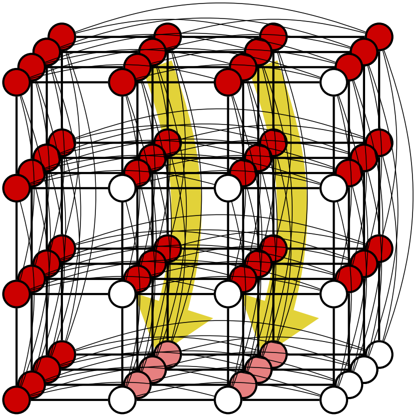


One set to force them all







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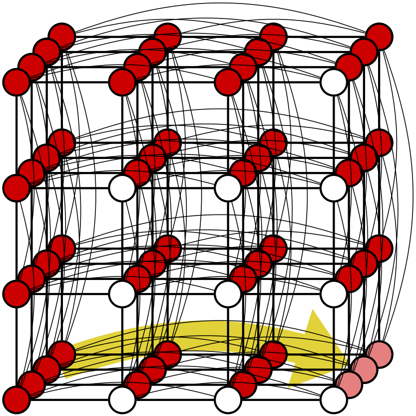

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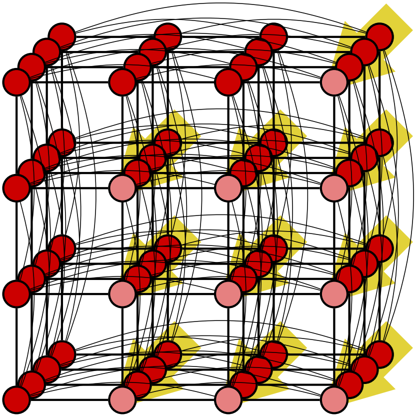

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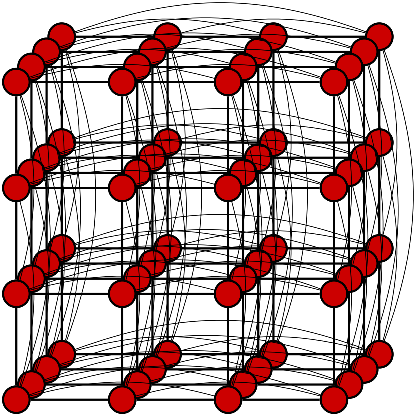

$$Z(H(n,q)) \leq \frac{1}{2}(q^n + (q-2)^n)$$

A matrix M *represents* a graph G if

$$x \sim y \quad \Leftrightarrow \quad (M)_{xy} \neq 0$$

for every two distinct vertices x, y.

The bound is tight

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Choose $M = A + I \implies Z(H(n,q)) \geq \frac{1}{2}(q^n + (q-2)^n)$

Our result

Theorem (Abiad, Simoens, Zeijlemaker, 2024)

$$Z(H(n,q)) = \frac{1}{2}(q^n + (q-2)^n)$$

Thank you for listening!

A. Abiad, R. Simoens and S. Zeijlemaker, *On the diameter and zero forcing number of some graph classes in the Johnson, Grassmann and Hamming association scheme*, Discrete Appl. Math. **348** (2024) 221-230.